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A Generalization of the Varisolvency and Unisolvency Properties

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1. INTRODUCTION

The theory of approximation by nonlinear unisolvent families goes back to Motzkin [8, 9] and Tornheim [16, 17]. For such families one has theorems quite analogous to the standard existence, uniqueness, and characterization theorems of linear Chebyshev theory. Later, Rice [14, 15] introduced the more general concept of varisolvent families and obtained uniqueness and characterization theorems. In this paper we extend the concepts of unisolvence and varisolvence to cover cases where all members of a given approximation family are constrained to pass through certain points.

We were motivated in this study by several considerations. First, we were interested in extending our previous work [5] on transformations of approximating families to the case where the transformation function $W(x, y)$ is of the form xy, or anything similar where the monotonicity requirement on W as a function of y does not hold for some x. It seems reasonable that by making a clever choice of the transformation function (guided by the characteristics of the function or data to be approximated), a very good fit can be achieved in many cases. Second, a generalization of this sort would have immediate application to the problem of approximation with simultaneous interpolation. A related problem which arises frequently in fitting a theoretical curve to an experimental one, namely that of finding a best approximation to a given curve from functions which pass through some specified point or points, also fits into our theory; in fact, general Hermite constraints may be treated in certain cases.

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In Section 2 of this paper we define the appropriate generalization of varisolvency and, by correspondingly modifying the usual definitions of curve intersections, "alternants," etc., we arrive at a characterization and uniqueness theorem analogous to the usual one. In Section 3 we carry out a similar program for unisolvency, and follow this by (in Section 4) the desired results on transformations of approximating families. Section 5 then contains a collection of miscellaneous theorems, including a continuity theorem and the expected application of the theory to problems involving simultaneous interpolation. Finally, in Section 6 we present an algorithm for and some remarks on the actual computation of best approximations in the context of our theory.

2. S-Varisolvent Families

First we need to define the concepts under consideration. Throughout this paper we will be dealing with continuous real-valued functions defined on a closed interval X of the real line.

Definition 1. Let $\{x_i\}_{i=1}^m$ be m distinct points in X and let $\{k_i\}_{i=1}^m$ be real numbers. Then a family F of continuous functions on X is called an S-family with respect to $\{(x_i, k_i)\}_{i=1}^m$ if $f(x_i) = k_i$ ($i = 1,..., m$) for every f in F. Each x_i is designated as either a "minus point" or a "plus point." (This designation is quite arbitrary; however, we shall see that in applying the theory there is usually a natural way to make the distinction.)

The sets $\{x_i\}_{i=1}^m$ and $X - \{x_i\}_{i=1}^m$ will play a key role in everything that follows and, for brevity, will be referred to as X_s and X', respectively.

Definition 2. A function $g \in C[X]$ is said to have an S-zero at $x^* \in X$ (relative to X_s) if $g(x^*) = 0$ and one of the following conditions holds:

(a) $x^* \in X'$.

(b) x^* is a minus point which is not an endpoint of X and g does not change sign at x^*.

(c) x^* is a plus point which is not an endpoint of X and g does change sign at x^*.

If (a) holds, x^* is not an endpoint of X, and g does not change sign at x^*, x^* is said to be an S-zero of multiplicity two; in all other cases, x^* is called an S-zero of multiplicity one.

Definition 3. An S-family F is said to satisfy property $S - Z$ of degree n at f_1 in F if, for all $f \in F, f \not\equiv f_1 \Rightarrow f - f_1$ has at most $n - 1$ S-zeros.

DEFINITION 4. An S-family F is said to be S-solvent of degree n at $f_1 \in F$ if, given any $\epsilon > 0$ and distinct points $\{x_j'\}_{j=1}^n \subset X'$, there exists a $\delta = \delta(f_1, \epsilon, x_1', \ldots, x_n') > 0$ such that if $| y_j - f_1(x_j')| < \delta$ for all j then there is an f_2 in F such that $f_2(x_j') = y_j$ for all j and $\| f_1 - f_2 \| < \epsilon$. (We are, of course, using the uniform or Chebyshev norm in this paper.)

DEFINITION 5. An S-family F is said to be S-varisolvent if for each $f \in F$ there is a finite degree $n = n(f)$ such that F satisfies property $S - Z$ of degree n at f and is S-solvent of degree n at f.

We note that the degree of varisolvence is uniquely determined at each f; this can be shown by a straightforward contradiction argument.

In order to characterize best approximations from S-varisolvent families, we must introduce a modified definition of the concept of alternation. Let $\{x_j'\}_{j=0}^r$ be a set of points in X' satisfying $x_j' < x_{j+1}'$ $(j = 0, 1, \ldots, r-1)$.

DEFINITION 6. A function $g \in C[X]$ is said to S-alternate in sign on $\{x_j'\}_{j=0}^r \subseteq X'$ if $\operatorname{sgn} g(x_j') = (-1)^{j+\sigma_j} \operatorname{sgn} g(x_0')$ for all j, where σ_j is the number of minus points in $[x_0', x_j']$. Equivalently, g takes on different signs at two adjacent points of $\{x_j'\}_{j=0}^r$ if the number of minus points between the two points is even, and g takes on the same sign otherwise.

DEFINITION 7. A function $g \in C[X]$ is said to have n S-alternations on X if there exists $\{x_j'\}_{j=0}^n \subseteq X'$ such that g S-alternates in sign on $\{x_j'\}_{j=0}^n$ and $| g(x_j')| = \| g \|$ for all j.

The following simple lemma, which follows directly from the definitions, shows why the modified definition of alternation is useful.

LEMMA 1. *If f_1 and f_2 belong to an S-family F, and if $f_1 - f_2$ S-alternates in sign on $\{x_0', x_1'\}$, then $f_1 - f_2$ has an S-zero in (x_0', x_1').*

The next lemma is a natural extension of a lemma proved by Rice [15] for the case of ordinary varisolvence.

LEMMA 2. *Let f_1 and f_2 be distinct members of an S-varisolvent family F, and let the degree at f_1 be n. Then $f_1 - f_2$ has at most $n - 1$ S-zeros counting multiplicity.*

Proof. Suppose the lemma is false. Let $\{x_j'\}_{j=1}^k$ be the distinct S-zeros of $f_1 - f_2$, and assume for concreteness that x_1' is an S-zero of multiplicity two, with $f_2 - f_1 > 0$ near x_1'. As in Rice's proof, since the degree of S-varisolvence at f_2 is at least $k + 1$ (by property $S - Z$), given $\epsilon > 0$ we can find an $f_3 \in F$ with $\| f_2 - f_3 \| < \epsilon$, $f_3(x_j') = f_2(x_j')$ (for all $j \neq 1$), and $f_3(x_1') < f_2(x_1')$. Taking ϵ sufficiently small, we see by examination of the possible types of S-zeros that $f_1 - f_3$ has at least as many S-zeros (counting multiplicity) as $f_1 - f_2$, and of these S-zeros at least $k + 1$ are distinct. Repeating this process as often as necessary, we finally obtain a function $f_l \in F$ such that $f_1 - f_l$ has at least n distinct S-zeros, which contradicts property $S - Z$.

Incidentally, Lemma 2 can still be proved, though with considerably more difficulty, if the requirement $\| f_1 - f_2 \| < \epsilon$ is omitted from the definition of S-varisolvence (see Definitions 4 and 5). A proof can be found in [6].

The next lemma is a very useful extension of a lemma due to Novodvorskii and Pinsker [12].

LEMMA 3. *Let g be a function continuous on X, and let $F \subseteq C[X]$ satisfy property $S - Z$, counting multiplicities, of degree n at $f_1 \in F$. If $f_1 - g$ S-alternates in sign on a point set $\{x_j'\}_{j=0}^n \subseteq X'$, then any function $f_2 \in F$ distinct from f_1 satisfies*

$$\max_j |f_2(x_j') - g(x_j')| > \min_j |f_1(x_j') - g(x_j')| .$$

Proof. Assume that the lemma is false, i.e., that there is an $f_2 \in F$ for which $\max_j |f_2(x_j') - g(x_j')| \leqslant \min_j |f_1(x_j') - g(x_j')|$. Let

$$\Delta(x) \equiv f_1(x) - f_2(x) = [f_1(x) - g(x)] - [f_2(x) - g(x)];$$

clearly $\Delta(x)$ S-alternates in sign (possibly vanishing at some points) on $\{x_j'\}_{j=0}^n$. We will prove by induction on n that $\Delta(x)$ has at least n S-zeros on $[x_0', x_n']$, which contradict property $S - Z$ and so proves the lemma. Notice first that the case $n = 1$ follows immediately from Lemma 1. Assume that the induction hypothesis holds for $n \leqslant N$. If $\Delta(x_J') \neq 0$ for some $J\,(1 \leqslant J \leqslant N)$, then there are at least J S-zeros of Δ on $[x_0', x_J']$ and at least $N + 1 - J$ more S-zeros on $[x_J', x'_{N+1}]$, giving a total of at least $N + 1$ S-zeros and completing the induction. Otherwise $\Delta(x_j') = 0$ for $j = 1, 2, \ldots, N$, and again the induction is completed unless these are the only S-zeros of Δ on $[x_0', x'_{N+1}]$ and they are all of multiplicity one. But this last possibility can not occur, since it would imply $\operatorname{sgn} \Delta(x'_{N+1}) = (-1)^{N+\sigma} \operatorname{sgn} \Delta(x_0')$, where σ is the number of minus points in (x_0', x'_{N+1}); and from this follows

$$\operatorname{sgn}[f_1(x'_{N+1}) - g(x'_{N+1})] = (-1)^{N+\sigma} \operatorname{sgn}[f_1(x_0') - g(x_0')],$$

which contradicts the S-alternation in sign of $f_1 - g$.

From Lemmas 2 and 3 we quickly obtain the following analog of the well-known result of de La Vallee Poussin.

COROLLARY 1. *Let F be S-varisolvent on X. If $f - g$ S-alternates in sign on $\{x_j'\}_{j=0}^n \subseteq X'$, where n is the degree of F at f, then*

$$\rho_F(g) \equiv \inf_{f_1 \in F} \| f_1 - g \| \geqslant \min_j | f(x_j') - g(x_j')| .$$

Proof. It follows from the lemmas that, for any f_1 in F,

$$\| f_1 - g \| \geqslant \max_j | f_1(x_j') - g(x_j')| \geqslant \min_j | f(x_j') - g(x_j')| ;$$

hence $\inf_{f_1 \in F} \| f_1 - g \| \geqslant \min_j | f(x_j') - g(x_j')|$, as was to be proven.

At this point we are able to prove an alternation theorem characterizing best approximations.

THEOREM 1. *Let g be a continuous function on X and let f be a member of an S-family F which is S-varisolvent on X and has degree n at f. Then:*

(i) *If $f - g$ has n S-alternations on X, then f is the unique best approximation to g from F.*

(ii) *If X_s is nonempty, f is a best approximation to g from F, and $\rho_F(g) > \max_i | g(x_i) - k_i |$, then $f - g$ has n S-alternations on X, and f is the unique best approximation.*

Proof. (i) Suppose that $f - g$ S-alternates on $\{x_j'\}_{j=0}^n$. Consider any $f_1 \in F$ such that $f_1 \neq f$. Then by Lemmas 2 and 3 we have that

$$\| f_1 - g \| \geqslant \max_j | f_1(x_j') - g(x_j')| > \min_j | f(x_j') - g(x_j')| = \| f - g \| .$$

(ii) The argument, which is a fairly straightforward extension of Rice's proof [15] of the analogous theorem in the varisolvency case, will be only sketched here. Suppose that $f - g$ S-alternates exactly k times, for $k < n$. Let $\alpha = x_0' < x_{n-k}' < \cdots < x_n' = \beta$ be a partitioning of $X = [\alpha, \beta]$ into $k + 1$ subintervals such that

(a) $| f(x_j') - g(x_j')| < \| f - g \|$ for $j = n - k, \; n - k + 1, \ldots, n - 1$, but $x_j' \notin X_s$ for these values of j; (b) $f - g$ has one S-alternation in any two adjacent subintervals, but none in any single subinterval. We assume that $| f(\alpha) - g(\alpha)| < \| f - g \|$; if this is not the case, similar arguments to those below will still work. (Notice that under our hypotheses the troublesome case of a constant error curve can not occur.) Choose $\delta > 0$ such that $| f(x) - g(x)| < \| f - g \|$ for all x in $[\alpha, \alpha + \delta]$, and pick a set of points $\{x_1', \ldots, x_{n-k-1}'\} \subseteq [\alpha, \alpha + \delta] - X_s$. Now pick any x^* where

$$| f(x^*) - g(x^*)| = \| f - g \|.$$

For any $\epsilon > 0$ we can find, by S-varisolvency and Lemma 2, an f_1 in F such that $f - f_1$ changes sign at x_j' ($j = 1,..., n - 1$), $|f_1(x^*) - g(x^*)| < \|f - g\|$, and $\|f - f_1\| < \epsilon$. For ϵ sufficiently small it can then be shown that f_1 is a better approximation to g than f, which is a contradiction. The uniqueness now follows from (i).

3. S-UNISOLVENT FAMILIES

This section provides a generalization of the unisolvency property similar to that of the varisolvency property discussed in the preceding section. The key definition is as follows.

DEFINITION 8. An S-family F is said to be S-unisolvent of degree n if property $S - Z$ holds with degree n for every $f \in F$ and, given any n distinct points $\{x_j'\}_{j=1}^n$ in X' and any n real numbers $\{y_j\}_{j=1}^n$, there exists an f in F such that $f(x_j') = y_j$ for all j.

The first theorem is a convergence theorem essential for later results.

THEOREM 2. *Let F be S-unisolvent of degree n. If the $2n$ sequences $\{x'_{1k}\},..., \{x'_{nk}\}$, $\{y_{1k}\},..., \{y_{nk}\}$ converge to $x_1',..., x_n'$, $y_1,..., y_n$, respectively, where all the y's are in R, all the points x' are in X', and $x_i' < x'_{i+1}$ for all i, then the sequence of functions f_k in F determined by (x'_{ik}, y_{ik}) ($i = 1,..., n$) converges uniformly to the function $f \in F$ passing through $(x_1', y_1),..., (x_n', y_n)$.*

The proof, which is an extension of a proof due to Tornheim [16], is by contradiction. The details will be omitted here, but the general idea is the following. Supposing the theorem is false, we can find, for any $\epsilon > 0$, a subsequence of $\{f_j\}_{j=1}^\infty$ (call it $\{f_j\}$ for simplicity) and a corresponding sequence $\{\xi_j\}_{j=1}^\infty \subseteq X$ such that $|f_j(\xi_j) - f(\xi_j)| > \epsilon$ for all j, and $\xi_j \to \xi \in X$. Without loss of generality we may assume that $\xi_j \downarrow \xi$ and that $f_j(\xi_j) - f(\xi_j)$ takes on the same sign for all j. We then divide the proof into various cases depending on the location of the points $x_1',..., x_n'$, ξ; in each case it is possible to construct a function $f^* \in F$ such that, for j sufficiently large, $f^* - f_j$ has too many S-zeros.

The above theorem has as an immediate consequence the following corollary from which we can conclude that Theorem 1 is also applicable to S-unisolvent families.

COROLLARY 2. *If F is S-unisolvent, then F is S-varisolvent.*

As is well known for varisolvent families, best approximations from S-varisolvent families do not necessarily exist. This is not the case for S-unisolvent families, as the following theorem shows.

THEOREM 3. *Let g be a continuous function on X and let the family F be S-unisolvent of degree n on X. Then a best approximation to g from F exists.*

Proof. Let $\{f_k\}$ be a sequence of functions in F such that $\|f_k - g\| \to \rho_F(g)$ as $k \to \infty$. Choose any set of n distinct points $\{x_j'\}_{j=1}^n$ in X'; then there exists a subsequence of $\{f_k\}$ whose values on $x_1',..., x_n'$ converge to real numbers $y_1,..., y_n$, respectively. By Theorem 2 this subsequence converges uniformly to the function $f \in F$ satisfying $f(x_j') = y_j$ for all j. Thus $\|f - g\| = \rho_F(g)$ and f is a best approximation.

4. TRANSFORMATIONS WHICH PRODUCE S-VARISOLVENT AND S-UNISOLVENT FAMILIES

In this section we consider ways of producing S-varisolvent and S-unisolvent families by carrying out transformations on simpler families. As in a previous paper [5] dealing with ordinary varisolvent and unisolvent families, the form of transformation that we shall consider is closely related to the "generalized weight function" studied by Moursund and others [10, 11]. We note that Dunham [4] has also pointed out the connection between weighted approximation and transformation of approximating families.

Since we are here working in the context of S-families, throughout the discussion $X_s \equiv \{x_i\}_{i=1}^m$ will be a fixed set of distinct points in X, and $\{k_i\}_{i=1}^m$ will be a fixed set of real numbers. Then we consider functions $W(x, y)$ mapping $X \times R$ into R and satisfying the following properties:

(A) $W(x, y)$ is continuous on $X \times R$.

(B) $W(x_i, y) = k_i$ for $i = 1,..., m$ and for all y.

(C) If $x^* \in X'$, $W(x^*, y)$ is a strictly monotonic function of y.

It is now necessary to discuss how the distinction between plus points and minus points is to be made. If x_i is not an endpoint of X, and if, for all sufficiently small $\delta > 0$, one of $W(x_i - \delta, y)$, $W(x_i + \delta, y)$ is an increasing function of y and the other is a decreasing function of y, then we call x_i a minus point of $W(x, y)$. If x_i is an endpoint of X, or if, for all sufficiently small $\delta > 0$, $W(x_i - \delta, y)$ and $W(x_i + \delta, y)$ are both increasing or both decreasing functions of y, then we call x_i a plus point of $W(x, y)$.

EXAMPLES. Let $X = [-1, 1]$. Then $W(x, y) = xy$ has a minus point at $x = 0$, while $W(x, y) = \exp(x^2y)$ has a plus point at $x = 0$.

The following lemma shows that there is no point of X_s which is not either a plus point or a minus point.

LEMMA 4. *Let $\bar{X}$ be a closed subinterval of X containing no points of X_s. Then $W(x^*, y)$ is either an increasing function of y for all x^* in $\bar{X}$ or a decreasing function of y for all x^* in $\bar{X}$.*

Proof. Suppose that the lemma is false. Then for any $y_1 \neq y_2$ the function $h(x) \equiv W(x, y_1) - W(x, y_2)$ takes on both positive and negative values in $\bar{X}$, because of (C). Then, by (A), $h(x') = 0$ for some x' in $\bar{X}$; but this contradicts (C).

We now prove the first transformation theorem.

THEOREM 4. *Let $W(x, y)$ satisfy conditions* (A), (B), *and* (C), *and let F_1 be a varisolvent family on X. Let $F \equiv \{W(x, f(x)) : f \in F_1\}$, and let the plus and minus points of F be the plus and minus points, respectively, of $W(x, y)$. Then F is S-varisolvent on X, and the degrees of the members of F_1 are not altered by the transformation.*

Remark. We need not include a separate definition of varisolvency here, since a varisolvent family is simply an S-varisolvent one where the set of plus and minus points is empty. If this set is not empty, but is contained in the set of plus and minus points of $W(x, y)$, a transformation theorem can still be proved. We note here that an analogous remark will hold for our later discussion of transformations of unisolvent families.

Proof of Theorem 4. We shall first use a contradiction argument to show that property $S - Z$ carries over from F_1 to F. Assume that there are functions f_1 (at which F_1 has degree n) and f_2 in F_1 such that $W(x, f_1(x)) - W(x, f_2(x))$ has S-zeros at the distinct points $x_1', \ldots, x_n'$ in X. If $x_j' \in X'$, then $f_1(x_j') = f_2(x_j')$ by property (C). If x_j' is a minus point, then again $f_1(x_j') = f_2(x_j')$, for otherwise $f_2(x) - f_1(x)$ will have the same (nonzero) sign throughout a neighborhood of x_j'. It then follows from the definition of a minus point of W that $W(x, f_1(x)) - W(x, f_2(x))$ will change sign as x passes through x_j', but this contradicts the definition of an S-zero. A similar argument holds in case x_j' is a plus point, so $f_1(x_j') = f_2(x_j')$ for $j = 1, \ldots, n$. Therefore $f_1 \equiv f_2$, and $W(\cdot, f_1) \equiv W(\cdot, f_2)$.

In order to show that $\{W(\cdot, f) : f \in F_1\}$ is S-solvent of degree n at f_1, let $\epsilon > 0$ and distinct points $x_1', x_2', \ldots, x_n' \in X'$ be given. Let

$$I = [\min_{x \in X} f_1(x) - \epsilon, \ \max_{x \in X} f_1(x) + \epsilon].$$

Since $W(x, y)$ is uniformly continuous on the compact set $X \times I$, we can find an ϵ^* satisfying $0 < \epsilon^* \leqslant \epsilon$ and such that $| l_1 - l_2 | < \epsilon^*$ implies $| W(x, l_1) - W(x, l_2)| < \epsilon$ for all $x \in X$ and all l_1, l_2 in I. Thus for any function g on X,

$$\| g - f_1 \| < \epsilon^* \Rightarrow \| W(\cdot, g) - W(\cdot, f_1)\| < \epsilon. \tag{1}$$

Then by the varisolvence of F_1 at f_1 we can find $\delta^* > 0$ such that

$$|\alpha_j - f_1(x_j')| < \delta^* \qquad \text{for all } j \Rightarrow \tag{2}$$

$\{$there exists an $f_2 \in F_1$ such that $f_2(x_j') = \alpha_j$ for all j and $\|f_1 - f_2\| < \epsilon^*\}$.

Now for each $x \in X'$, $W_x(y) \equiv W(x, y)$ is a continuous, strictly monotonic function of y with a continuous, strictly monotonic inverse W_x^{-1}. From (A), (C), and the continuity of $W_{x_j'}^{-1}$ for all j we can find a number $\delta > 0$ such that for every j $|y_j - W(x_j', f_1(x_j'))| < \delta$ implies that y_j is in the range of $W_{x_j'}$ (i.e., $W_{x_j'}^{-1}(y_j)$ exists) and $|W_{x_j'}^{-1}(y_j) - f_1(x_j')| < \delta^*$. Then it follows from (2) that there is an $f_2 \in F_1$ such that $f_2(x_j') = W_{x_j'}^{-1}(y_j)$ for all j and $\|f_1 - f_2\| < \epsilon^*$. Therefore $W(x_j', f_2(x_j')) = y_j$ for all j and, by (1), $\|W(\cdot, f_2) - W(\cdot, f_1)\| < \epsilon$. This completes the proof of the theorem.

The following similar theorem holds for S-unisolvent families. We will omit the proof, since the proof of property $S - Z$ is the same as in Theorem 4, and the rest of the proof is straightforward.

THEOREM 5. *Let $W(x, y)$ satisfy properties* (A), (B), (C), *and also* (D): $\lim_{|y|\to\infty} |W(x, y)| = \infty$ *for all x in X'. Let F_1 be a unisolvent family of degree n on X. If $F \equiv \{W(x, f(x)) : f \in F_1\}$, and if the plus and minus points of F are defined to be the plus and minus points, respectively, of $W(x, y)$, then F is S-unisolvent of degree n on X.*

Example. Let $X = [-1, 1]$, $W(x, y) \equiv xy$, and $F_1 \equiv P_0$, the polynomials of degree zero. Then the family $F \equiv \{cx : c \in R\}$ has a minus point at $x = 0$ and is S-unisolvent of degree one on X. Suppose we wish to find a best approximation from F to the function $g(x) \equiv e^x$. Results of the preceding sections show that there exists a unique best approximation and it is characterized by the S-alternation property (Theorem 1). We readily compute this best approximation to be $f(x) = (e^2 - 1)\, x/2e$. Note that

$$\|f - g\| = |g(-1) - f(-1)| = |g(1) - f(1)| = (e^2 + 1)/2e.$$

5. FURTHER PROPERTIES OF S-VARISOLVENT FAMILIES

S-varisolvent (and S-unisolvent) families can arise in ways other than by transformations, as the following theorem shows.

THEOREM 6. *Let F_1 be a varisolvent family on X. Let $\{x_i\}_{i=1}^m$ be a set of distinct points on X, and $\{y_i\}_{i=1}^m$ be real numbers. Suppose that the family $F \equiv \{f \in F_1 : f(x_i) = y_i\,;\, i = 1, 2,..., m\}$ has more than one element. Consider F as an S-family with respect to $X_s = \{x_1 ,..., x_m\}$, where all points of X_s are* *designated as minus points. Then F is S-varisolvent, and, for every f in F, the degree of F at f is m less than the degree of F_1 at f.*

Proof. Let $D_1(f)$ be the degree of F_1 at f and define $D(f) = D_1(f) - m$ for all f in F. We first note that $D(f)$ is always positive, for otherwise we would have $D_1(f) \leqslant m$ for some $f \in F$ and F could have at most one member, because F_1 satisfies property Z. Now consider any $f \in F$ and let $D(f) = l$. We wish to prove that F is S-solvent of degree l at f. Let $\epsilon > 0$ and distinct points $x_1',..., x_l'$ in X' be given. Since $D_1(f) = m + l$, there exists a $\delta(\epsilon, f, x_1 ,..., x_m , x_1' ,..., x_l') > 0$ such that if $| f(x_j') - y_j' | < \delta$, for $j = 1,..., l$, then (applying varisolvency of F_1 and noting that $| f(x_i) - y_i | = 0 < \delta$, for $i = 1,..., m$) there is an f_1 in F_1 such that $f_1(x_i) = y_i$ ($i = 1,..., m$), $f_1(x_j') = y_j'$ ($j = 1,..., l$), and $\| f - f_1 \| < \epsilon$. But we note that this f_1 is also in F, and its properties are just those needed to demonstrate the desired degree of S-solvency at f. To prove property $S - Z$, suppose there is an f_1 in F distinct from f and such that $f - f_1$ has $l_1 \geqslant l$ S-zeros, t of them occurring in X_s. By definition, an S-zero at a minus point is a multiplicity-two zero in the ordinary sense. Thus the number of zeros of $f - f_1$ in the ordinary sense is at least $(m - t) + 2t + (l_1 - t) = m + l_1 \geqslant m + l = D_1(f)$, which contradicts property Z on F_1.

Theorem 6 has immediate applicability to the problem of finding a function f from F which best approximates a given function g and satisfies the additional conditions $f(x_i) = g(x_i)$, $i = 1,..., m$. From the theorems of the previous sections we then readily recover the results of Barrar and Loeb [1] on best approximation with simultaneous interpolation.

Furthermore, if we are dealing with approximation from a family of the form $\{W(x, p(x)) : p \in P_n\}$, where P_n is the family of all polynomials of degree at most n and $W(x, y)$ satisfies conditions (A), (B), (C) of Section 4, we can use the following approach to treat Hermite constraints. Let $\bar{x}_1 ,..., \bar{x}_t$ be a set of distinct points of X', and associate with each point $\bar{x}_i$ a positive integer $r(i)$ such that $l = \sum_i r(i) \leqslant n + 1$. Given any continuous function g on X and any real numbers a_{ki} ($k = 0,..., r(i) - 1$ and $i = 1,..., t$), the problem is to find the best approximation to g from the set

$$F = \{W(x, p(x)) : p \in P_n , (d^k/dx^k) W(x, p(x))|_{x=\bar{x}_i} = a_{ki} \\ \text{for } k = 0,..., r(i) - 1 \text{ and } i = 1,..., t\}.$$

Assume that $\partial^M W(x, y)/\partial x^s \partial y^{M-s}$ is continuous on $X' \times (-\infty, \infty)$ for all s with $0 \leqslant s \leqslant M$ [where $M = \max_i (r(i) - 1)$], and that $\partial W(x, y)/\partial y \neq 0$ throughout $X' \times (-\infty, \infty)$ if $M \geqslant 1$. Suppose further that $W_{\bar{x}_i}^{-1}(a_{0i})$ exists for $i = 1,..., t$. Introduce the transformation function

$$W^*(x, y) \equiv W(x, (x - \bar{x}_1)^{r(1)} (x - \bar{x}_2)^{r(2)} \cdots (x - \bar{x}_t)^{r(t)} y + p^*(x)),$$

where p^* is an arbitrary member of P_n such that $W(x, p^*(x)) \in F$. (The existence of such a p^* is shown in [6] in a more general setting.) It can now be shown that $F = \{W^*(x, p(x)) : p \in P_{n-l}\}$ (where $P_{-1} \equiv \{0\}$), and that this family is S-varisolvent. Furthermore, F will be S-unisolvent of degree $n - l$ if $W(x, y)$ also satisfies (D). All the results of our theory can thus be applied. Notice that the plus and minus points of F (or of W^*) include those of W in addition to $\bar{x}_1, \ldots, \bar{x}_t$. We can now combine approximation and Hermite interpolation of a given function g by taking $a_{ki} = g^{(k)}(\bar{x}_i)$ for $k = 0, \ldots, r(i) - 1$ and $i = 1, \ldots, t$.

The problem of approximation with Hermite constraints has also been studied by Loeb et al. [7] in the rather different context of weighted approximation from certain linear subspaces (with the restriction $a_{0i} = g(\bar{x}_i)$, $i = 1, \ldots, t$), and by Perrie [13] for rational approximation.

Another problem of considerable practical importance is that of approximating a given function on a finite point set. The following theorem, which is a generalization of Theorem 1, is useful in this connection. The proof, which just involves slight modifications of that of Theorem 1, will be omitted.

THEOREM 7. *Let X be a closed interval and let X_1 be any compact subset of X containing at least $n + 1$ points. Let the family F be S-varisolvent on X with degree n at f. Suppose g is any function continuous on X. Then*: (i) *If f is a best approximation on X_1 to g from F and if*

$$\| g - f \|_{X_1} > \max_{x_i \in X_s \cap X_1} | g(x_i) - k_i |$$

(*where the* max *over the empty set is taken to be zero*), *and if $(X - X_1) \cup X_s$ is not empty, then $g - f$ has at least n S-alternations on X_1* (i.e., *the points of alternation, where $| g - f | = \| g - f \|_{X_1}$, lie in X_1 whether the plus and minus points do or not*). *Furthermore f is the unique best approximation.*

(ii) *If $g - f$ has n S-alternations on X_1, then f is the unique best approximation to g on X_1.*

We will end this section by presenting a continuity theorem. The proof, which may be found in [6], follows the lines indicated by Dunham [3] for the case of ordinary varisolvency and will be omitted here. To simplify the statement of the theorem, let $T_F(g)$ denote the best approximation (if one exists) to a function g from a family F.

THEOREM 8. *Let F be S-varisolvent and let the maximal degree of F be n. Let g be any continuous function on X. Let $f = T_F(g)$ be a function at which the degree is n, and suppose that $\rho_F(g) > \max_{x_i \in X_s} | g(x_i) - k_i |$. Then there exists a $\delta > 0$ such that $T_F(g_1)$ exists for any $g_1 \in C[X]$ satisfying*

$\| g - g_1 \| < \delta$. *Furthermore, if a sequence* $\{g_k\}$ *converges to* g *uniformly, then the corresponding sequence* $\{T_F(g_k)\}$ *converges to* $T_F(g)$ *uniformly.*

6. An Algorithm for Construction of Best Approximations

A Remez-type algorithm can be used to find best approximations from S-unisolvent families. The algorithm which we discuss in this section is a single-exchange method very similar to that given by Novodvorskii and Pinsker [12] for unisolvent families. Before presenting the algorithm and proving its convergence, we need a preliminary lemma.

Lemma 5. *Let* F *be an* S*-unisolvent family of degree* n *on* X, *and let* g *be an arbitrary continuous function on* X. *For any arbitrary set of* $n + 1$ *distinct points* $\{x_j'\}_{j=0}^n \subseteq X'$, *the system of equations*

$$f(x_j') - g(x_j') = (-1)^{j+\sigma_j} \tilde{E}, \qquad j = 0,..., n, \tag{3}$$

where σ_j *is the number of minus points between* x_0' *and* x_j', *has a unique solution* $(f, \tilde{E})$ *with* $f \in F$ *and* $\tilde{E} \in R$. *Furthermore, this solution is uniformly continuous in the points* $\{x_j'\}_{j=0}^n$.

Proof. First we prove existence; uniqueness will then follow from Lemma 3. Let f_0 be the unique function in F which satisfies

$$f_0(x_k') = g(x_k') \qquad \text{for} \quad k = 1,..., n.$$

We may assume that $f_0(x_0') - g(x_0') \neq 0$, since otherwise f_0 itself is a solution of (3). Without loss of generality, assume that $\Delta_0{}^0 \equiv f_0(x_0') - g(x_0') > 0$. Consider the functions $f_\lambda \in F$ defined by

$$f_\lambda(x_j') - g(x_j') = \Delta_0{}^0 \lambda (-1)^{j+\sigma_j} \qquad \text{for} \quad j = 1,..., n.$$

We see that, for positive λ, $f_\lambda(x_0') - g(x_0') - \Delta_0{}^0 < 0$, since otherwise the difference $f_\lambda - f_0 \equiv f_\lambda - g - (f_0 - g)$ will have at least n S-zeros in $[x_0', x_n']$, which contradicts property $S - Z$. Now define

$$h(\lambda) \equiv \lambda \Delta_0{}^0 - [f_\lambda(x_0') - g(x_0')].$$

Since $h(1) > 0$ and $h(0) = -\Delta_0{}^0 < 0$, by continuity there must be some $\bar{\lambda}$ in (0, 1) for which $h(\bar{\lambda}) = 0$, and $(f_{\bar{\lambda}}, \Delta_0{}^0 \bar{\lambda})$ is then the desired solution of (3). The proof of the uniform continuity will be omitted; it is a straightforward contradiction argument involving property $S - Z$.

Remark. It can be shown, again by invoking property $S - Z$, that $h(\lambda)$ varies monotonically as λ goes from zero to one. This fact is of practical importance in programming the algorithm for a computer.

We now present our algorithm. As in the preceding lemma, we assume that F is an S-unisolvent family of degree n on $X = [\alpha, \beta]$, and we let g be an arbitrary continuous function on X.

ALGORITHM. Begin by choosing a set of distinct points $X_0 = \{x'_{j0}\}_{j=0}^{n}$ in X'. Now find a function $f_0 \in F$ such that

$$|f_0(x'_{j0}) - g(x'_{j0})| = E_0 \quad (j = 0, 1, \ldots, n),$$

and $f_0 - g$ S-alternates in sign on X_0. Let $x^0 \in X$ be a point where $|f_0(x^0) - g(x^0)| = \|f_0 - g\|$. If x^0 is a point of X_s or of X_0, the algorithm is terminated. Otherwise, replace one point of X_0 by x^0 in such a way that $f_0 - g$ S-alternates in sign on the new set of points, which we designate by $X_1 = \{x'_{j1}\}_{j=0}^{n}$. Now repeat this process using X_1 instead of X_0 to obtain a function f_1 and, if the process does not terminate, a third point set X_2. Continuing in this way, we generate a finite or infinite sequence of functions $\{f_l\} \subseteq F$, together with corresponding quantities E_l and point sets $X_l = \{x'_{jl}\}_{j=0}^{n}$.

THEOREM 9. *Let F be S-unisolvent of degree n on X. If the above algorithm terminates at some f_j, then f_j is a best approximation to g from F. If the algorithm does not terminate, and if, for some l,*

$$E_l > \max_{i=1,\ldots,n} |g(x_i) - k_i|,$$

then the sequence $\{f_l\}$ converges uniformly to the best approximation to g from F.

The proof for the case when the algorithm terminates is an easy consequence of Theorem 1 and the definitions. If the algorithm does not terminate, the proof proceeds as follows. (Details may be found in [6].) First one proves that if $(x_0', \ldots, x_n') \in X^{n+1}$ is any limit point of the sequence $\{(x'_{0l}, \ldots, x'_{nl})\}_{l=1}^{\infty}$, then all of the points x_j' are distinct and none is in X_S. One can then show that the function constructed on such a limit point by Lemma 5 is a best approximation, and the sequence $\{f_l\}$ converges uniformly to it.

Remark. If $g(x_i) = k_i$ $(i = 1, \ldots, m)$ (i.e., the S-family interpolates the function to be approximated), then either the algorithm terminates or the condition that $E_l > \max_i |g(x_i) - k_i|$ for some l is automatically satisfied for $l = 2$.

The difficulty which arises in construction of best approximations from S-varisolvent families is that such approximations need not exist. To conclude this paper, we present a condition under which existence does hold and the algorithm above may be applied.

DEFINITION 9. Let F be an S-varisolvent family, and g be a continuous function on X. Define the "nth degree radius of S-unisolvence of g with respect to F on X" by

$$\begin{aligned}\mathrm{RSU}_X(n, F, g) \equiv \sup \{d\colon & \text{ for any distinct } x_1', \ldots, x_n' \in X', \\ & | g(x_i') - y_i | < d \text{ for } i = 1, \ldots, n \\ & \Rightarrow \exists f \in F \text{ of degree } n \text{ with } f(x_i') = y_i \ (i = 1, \ldots, n)\}.\end{aligned}$$

EXAMPLE. Let $X = [-1, 1]$, $F = \{e^{xy} : y \in P_1\}$, and $g(x) = x^2 + 1$. Then it can be readily seen that $\mathrm{RSU}_{[-1,1]}(2, F, g) = 1$.

The following theorems can then be shown to hold. (Proofs may be found in [6].)

THEOREM 10. *Let g be a continuous function on X, and let F be an S-varisolvent family on X. Let $X_1 \subseteq X$ contain at least n points of X'. Then if $\rho_F(g) < \mathrm{RSU}_{X_1}(n, F, g)$, there exists a best approximation to g from F.*

THEOREM 11. *Let F be an S-varisolvent family on X and let g be any continuous function on X. Then, with the additional hypothesis that for some n $\rho_F(g) < \mathrm{RSU}_X(n, F, g)$, Lemma 5 and Theorem 9 still hold.*

We have in fact carried out successful computer tests of the algorithm for both S-unisolvent and S-varisolvent families. Notice that although $\rho_F(g)$ is generally not known, a rough upper bound on that quantity is often all that is needed to show that the algorithm is applicable.

We also have a multiple-exchange algorithm (with the same convergence theorem), but from computer tests it appears that the increased programming complexity required may make any gain in speed negligible for many problems.

It should be noted that Barrar and Loeb [2] have recently published an algorithm and a local convergence theorem for approximation from nonlinear families which satisfy the local and global Haar conditions—conditions which imply varisolvency. They also assume that the best approximation is "normal"; our concept of the RSU (Definition 9) provides an alternate approach to handling the difficulties which arise in the absence of such a hypothesis.

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